

# On the Singularity at $p = 0.5$ in 1D Random Walk

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May 7, 2026

## 1 Introduction

In the classic 1D Random Walk (or Gambler's Ruin) problem, we track a particle on a line of states  $\{0, 1, \dots, N\}$ . At each step, it moves right with probability  $p$  and left with  $1 - p$ . The probability  $\alpha(i)$  of eventually reaching state  $N$  starting from  $i$  is governed by the boundary conditions  $\alpha(0) = 1$  and  $\alpha(N) = 0$ .

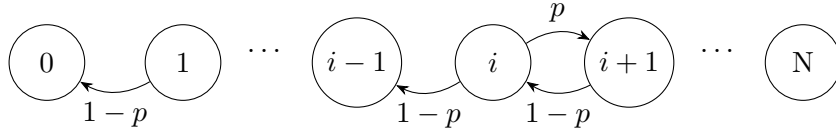


Figure 1: Markov Chain representation of the 1D Random Walk.

The system obeys the following recurrence relation for  $1 \leq i \leq N - 1$ :

$$\alpha(i) = p\alpha(i + 1) + (1 - p)\alpha(i - 1) \quad (1)$$

## 2 The Mathematical Conflict

The solution is derived from the partial sums of the ratio  $r = \frac{1-p}{p}$ . The explicit form for  $\alpha(i)$  is:

$$\alpha(i) = 1 - \frac{\sum_{k=0}^{i-1} r^k}{\sum_{k=0}^{N-1} r^k} = 1 - \frac{1 + r + r^2 + \dots + r^{i-1}}{1 + r + r^2 + \dots + r^{N-1}} \quad (2)$$

Using the geometric series formula for  $r \neq 1$ , this simplifies to:

$$\alpha(i) = 1 - \frac{(1 - r^i)/(1 - r)}{(1 - r^N)/(1 - r)} = \frac{r^i - r^N}{1 - r^N} \quad (3)$$

At  $p = 0.5$ , we have  $r = 1$ . Directly substituting  $r = 1$  into Eq. (3) yields the indeterminate form  $\frac{0}{0}$ . However, looking at the summation in Eq. (2) reveals a different story.

## 3 Some Observations

### 3.1 Direct Computation via Summation

The most elegant resolution is to realize that when  $p = 0.5$  ( $r = 1$ ), the geometric series in Eq. (2) doesn't "break"—it simply becomes a series of 1s. Each term  $r^k$  becomes  $1^k = 1$ :

$$\alpha(i) \Big|_{r=1} = 1 - \frac{\underbrace{1 + 1 + \dots + 1}_{i \text{ terms}}}{\underbrace{1 + 1 + \dots + 1}_{N \text{ terms}}} = 1 - \frac{i}{N} = \frac{N - i}{N} \quad (4)$$

This confirms that the "arithmetic" solution for  $p = 0.5$  is actually embedded within the "geometric" structure of the general solution.

### 3.2 Continuity via L'Hôpital's Rule

Treating the formula as a function of  $r$  and taking the limit  $p \rightarrow 0.5$ , where  $r \rightarrow 1$ :

$$\lim_{r \rightarrow 1} \frac{r^i - r^N}{1 - r^N} = \lim_{r \rightarrow 1} \frac{ir^{i-1} - Nr^{N-1}}{-Nr^{N-1}} = \frac{i - N}{-N} = \frac{N - i}{N} \quad (5)$$

This mathematical limit matches our direct summation result perfectly, proving the system is **continuous** even as the common ratio crosses the singularity at 1.

## 4 Interpretation

To understand why the solution form shifts, we can represent the recurrence as a first-order vector system. By defining a state vector  $\vec{v}_i = [\alpha(i), \alpha(i-1)]^T$ , we can rewrite Eq. (1) as a transition:

$$\begin{bmatrix} \alpha(i+1) \\ \alpha(i) \end{bmatrix} = \begin{bmatrix} \frac{1}{p} & -\frac{1-p}{p} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha(i) \\ \alpha(i-1) \end{bmatrix} \quad (6)$$

Let  $M$  be the transition matrix. The characteristic equation  $\det(M - \lambda I) = 0$  yields:

$$\lambda^2 - \frac{1}{p}\lambda + \frac{1-p}{p} = 0 \implies p\lambda^2 - \lambda + (1-p) = 0 \quad (7)$$

The eigenvalues are  $\lambda_1 = 1$  and  $\lambda_2 = r = \frac{1-p}{p}$ . The "Singularity" at  $p = 0.5$  is thus fundamentally a problem of **algebraic multiplicity**:

- **Asymmetric Case** ( $p \neq 0.5$ ):  $M$  has two distinct eigenvalues. The probability is a weighted sum of two independent geometric growth rates:  $\alpha(i) = c_1(1)^i + c_2(r)^i$ . The solution space is spanned by  $\{1^k, r^k\}$
- **Symmetric Case** ( $p = 0.5$ ): The eigenvalues collide at  $\lambda = 1$ . The matrix  $M$  becomes not diagonalizable, and the solution space collapse to  $span\{1, 1\}$ . To maintain a 2D solution space, we must introduce the term  $i \cdot 1^i$ , leading to the linear form.

Thus, the arithmetic progression  $\frac{N-i}{N}$  is the natural emergence of a **degenerate** geometric progression.

## 5 Conclusion

The transition at  $p = 0.5$  is more than a piecewise definition. It is a moment where the "Drift" (exponential power) vanishes, leaving only "Diffusion" (linear power). The 0/0 division is merely the mathematical signature of this symmetry being restored.

## 6 Acknowledgement

Solutions to the recursion (Eq. (2) and Eq. (3)) reference lecture 26 of CS 70 (Discrete Mathematics and Probability), Spring 2026, UC Berkeley ([www.eecs70.org/assets/pdf/slides/lec26.pdf](http://www.eecs70.org/assets/pdf/slides/lec26.pdf)), with slides credits by Professor Yun S. Song.